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FOREWORD

An understanding of the mathematical theory of finite elements becomes necessary when conducting research into advanced techniques for finite element application. Such an area of research exists in the development of contact/impact, finite element formulations presently being pursued at the Civil Engineering Laboratory (CEL). The work reported herein supports that project.

Much of the work presented was accomplished while the author was on leave for one academic year at the University of California, Santa Barbara, under the auspices of the CEL Fellowship Program. There, research was begun on the development of contact/impact, finite element formulation, but soon it became evident that a study of the mathematical theory of finite element was first necessary for a greater appreciation of element performance, convergence, and, in general, understanding why the finite element method works as it does.

An important outcome of the study is the belief that some important concepts of the mathematical theory of finite element can be presented to many practitioners without necessity for reference to higher mathematics because these concepts can be cast in the familiar geometrical notions of Euclidean space. This report was written and presented with that belief foremost in mind.

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The mathematical prerequisites from functional analysis that enable the understanding of the mathematical theory of finite element are organized, presented, and explained. They begin with the definition of linear vector spaces and include all intermediate definitions up to the definition of Hilbert spaces. The Ritz approximate solution method for boundary value problems is developed so that the close similarity between it and the finite element method can be observed. Solutions given by the Ritz method are projections of the true solution → next page

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CONTENTS

INTRODUCTION

The purpose of this report is to provide an explicative introduction to the mathematical theory of the finite element method. To really understand the finite element method today, it is necessary to know something of its mathematical foundations as they relate to accuracy, convergence, valid shape functions, and other mathematically based aspects which are associated with the method. Results from research into these questions are beginning to emerge and will soon impact the practitioner with greater frequency. The mathematical theory that has been established to date is not complete, but it is sufficient to understand both why the finite element method works and why it is efficient in the numerical solution of engineering boundary value problems.

The major obstacle in understanding is that the language must of necessity involve a minimum of functional analysis. This is a branch of mathematics with which most engineers are unfamiliar. To that end, the report begins with a section on prerequisite functional analysis. The intent is to present only enough information to enable most engineers to interpret the necessary functional analysis concepts within familiar, geometrical frameworks. These concepts are simple when viewed as generalizations of the ordinary three-dimensional Euclidian space notions of distance, dot products, Pythagorean theorem, orthogonal projection, and other concepts.

Mathematical respectability began to come to the finite element method when the unknown nodal point variables were recognized as the unknown coefficients of the classical Rayleigh-Ritz method. Thereafter, many of the initial attempts at clarifying the mathematical theory were presented in the framework of the Rayleigh-Ritz procedure. But soon it was realized that problems were being routinely solved by the finite

element method that could not also be formulated in a classical Rayleigh-Ritz way. The Rayleigh-Ritz method requires the existence of a variational formulation of a problem prior to its use in the problem's solution. Yet, many problems being solved had no such corresponding formulation.

For example, in structural mechanics most problems were being formulated on the basis of the principle of virtual work which is in fact more general than variational formulations. Therefore, the mathematical theory could not be completely developed from a classical Rayleigh-Ritz viewpoint.

From then on, the method's development was no longer the sole province of engineers who had originally developed the method and who indeed coined the phrase "finite element." Investigators who were familiar with variational methods and who also knew of a more general method - the classical Galerkin method - soon saw the advantage of thinking in terms of that method when seeking fundamental knowledge on the finite element method. Today, the finite element method is thought to be founded in an extension of the classical Ritz-Galerkin theory.

The finite element method, as a result, is now applicable to problems in fluid mechanics, heat conduction, electrostatics, and other areas, besides the traditional problems in elasticity and structural mechanics. In general, whenever the solution of a differential equation is sought within a region bounded by geometrically complex or "real" shapes, the finite element method is often more advantageous than other methods. The principal alternative method for the numerical solution of boundary value problems is the finite difference method.

SOME PREREQUISITES FROM FUNCTIONAL ANALYSIS

Functional analysis is the study of vector spaces resulting from a merger of geometry, linear algebra, and analysis. It serves as a basis for aspects of several important branches of applied mathematics, including

Fourier series, integral and differential equations, numerical analysis, and any field where linearity plays a key role. Its appeal as a unifying discipline stems primarily from its geometric character. Most of the principal results in functional analysis are expressed as abstractions of intuitive geometric properties of ordinary three-dimensional space.

Within the last 10 years, the finite element method, which is about 20 years old, has been given mathematical respectability. Gradually the method has been recognized as an extension of the Rayleigh-Ritz-Galerkin technique. To study the mathematical foundation of finite element it is first necessary to become comfortable with a certain number of concepts from functional analysis. Examples would include the concept of a norm and its interpretation as a natural measure of strain energy, and the identification of a Hilbert space as the collection of admissible functions in a physically derived variational principle.

The aim here is to present a minimum amount of functional analysis believed necessary to comprehend the mathematical foundations of the finite element method. We begin essentially from "ground zero," i.e., with the definition of a vector space and proceed up to the definition of a Hilbert space. Each succeeding definition builds upon the preceding definitions. Examples are given freely to aid the comprehension of the definition.

Many of the following definitions have been extracted from Luenberger.* Though the author does not address the subject of finite element, its treatment of functional analysis is well-suited to an introduction to the mathematical theory of finite element.

Linear Vector Space

A linear vector space X is a set of elements called vectors together with two operations

(1) Addition: given $x,y \in X$
then $x + y \in X$

*David G. Luenberger. Optimization by vector space methods. New York, N.Y., John Wiley and Sons, Inc., 1969.

(2) Scalar Multiplication:
given $x \in X$ and α any scalar
then $\alpha x \in X$

The set X and the above two operations are assumed to satisfy the following seven axioms.

(1) $x + y = y + x$

(2) $(x + y) + z = x + (y + z)$

(3) $x + \Theta = x$ (all $x \in X$ where Θ is defined as a null vector)

(4) $\alpha(x + y) = \alpha x + \alpha y$

(5) $(\alpha + \beta)x = \alpha x + \beta x$

(6) $(\alpha\beta)x = \alpha(\beta x)$

(7) $0x = \Theta$

Some examples of vector spaces are:

A. *Set of real numbers.* Addition and multiplication are defined in the usual way. The real number zero plays the role of the null vector. This space is denoted as R^1. *It is the one-dimensional real coordinate line.*

B. An n-dimensional real coordinate space R^n.

$x = (\zeta_1, \zeta_2, \ldots, \zeta_n)$

$\Theta = (0, 0, \ldots, 0)$

$y = (\eta_1, \eta_2, \ldots, \eta_n)$

$(x + y)_k = \zeta_k + \eta_k$

$(\alpha x)_k = \alpha\zeta_k$

C. Collection of all real-valued continuous functions on the interval [a,b] of the real line, C[a,b].

$\theta = f(t) = 0$ for all t on $[a,b]$

$(x + y)(t) = x(t) + y(t)$

$(\alpha x)t = \alpha x(t)$

Example C illustrates that a vector space may be composed of a set of continuous functions and therefore should not be thought of as being composed of "vectors per se" in all cases. Indeed, the vector spaces pertaining to finite element theory are composed of elements that are continuous functions.

Cartesian Product

Let X and Y be vector spaces over the same field of scalars. Then the Cartesian Product of X and Y, denoted X x Y, consists of the collection of ordered pairs (x,y) with $x \in X$ and $y \in Y$. The Cartesian product is a vector space with addition and multiplication defined as follows:

(1) Addition: $(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, y_1 + y_2)$

(2) Multiplication: $\alpha(x,y) = (\alpha x, \alpha y)$

These definitions satisfy the seven axioms of a linear vector space.

Subspace

A nonempty subset M of a vector space X is called a subspace of X if every vector of the form $\alpha x + \beta y$ is in M whenever x and y are both in M. Some examples of subspaces and subsets are shown in Figure 1 with respect to the R^2 vector space.

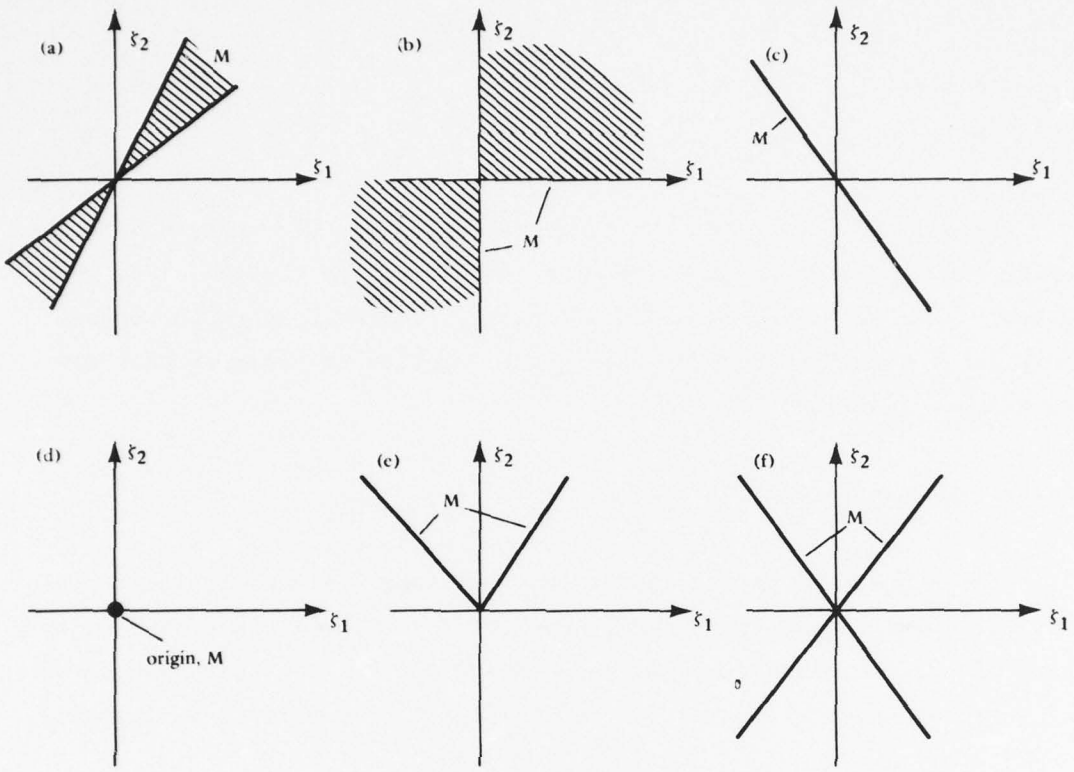

Figure 1. Subspaces and subsets.

Only examples (c) and (d) are subspaces of R^2. The others are merely subsets of R^2.

A subspace is itself a vector space. If it is not equal to the entire space, as in the above two examples, then it is a proper subspace. Examples of proper subspaces in R^3 include both a plane and a line running through the origin.

Norm

A norm is a measure of the size of an element belonging to a vector space or the distance between two elements in the space and is denoted by $\|x\|$ or $\|x - y\|$, respectively. If the vector space X has a norm defined on it, it becomes a normed vector space. This means that there

is a real valued function which maps each element x in X into a real number $\|x\|$ called the norm of x. The norm is assumed to satisfy the following three axioms:

(1) $\|x\| \geq 0$ for all $x \in X$, $\|x\| = 0$ if $x = \Theta$

(2) $\|\alpha x\| = \alpha \|x\|$ for all scalars α and all $x \in X$

(3) $\|x + y\| \leq \|x\| + \|y\|$ for each $x,y \in X$

The first axiom forces all elements, save the null element, to have positive measure. If the measure of something is zero ($\|x\| = 0$) then that something must be zero ($x = \Theta$) and conversely if something is zero ($x = \Theta$) then its measure should be zero ($\|x\| = 0$).

The second axiom states that x and its negative -x have the same measure, and that, for example, the measure of 3x is three times the measure of x.

The third axiom is the triangular inequality, so-called since the sum of the lengths of two sides of a triangle is never smaller than the length of the third side.

A norm is merely an abstraction of our usual concept of length. Some examples of norms follow. Some of the following real valued functions satisfy the three axioms, and some do not.

If $X = R^n$:

(a) $\|x\| = \sum_{i=1}^{n} \zeta_i$ cannot be used, violates axiom (1)

(b) $\|x\| = \sum_{i=1}^{n} |\zeta_i|$ is a valid norm

(c) $\|x\| = \sum_{i=1}^{n} \zeta_i^2$ cannot be used, violates axioms (2) and (3)

(d) $\|x\| = \max_{i=1,n} |\zeta_i|$ is a valid norm

(e) $\|x\| = \left(\sum_{i=1}^{n} |\zeta_i|^2 \right)^{1/2}$ is a valid and very useful norm called the Euclidean norm.

If $X = C[a,b]$:

(f) $\|x\| = \max_{a \leq t \leq b} |x(t)|$ is a valid norm and is the largest value of the function $x(t)$ on the interval $[a,b]$.

(g) $\|x\| = \int_a^b |x(t)| dt$ is a valid norm

(h) $\|x\|_o = \left(\int_a^b |x(t)|^2 dt \right)^{1/2}$

This is a valid norm and very useful in finite element theory. We give it the special designation H^o norm. This norm can be generalized to include the H^o norms of the functions and its derivatives. For example:

(i) $H^2 \equiv \|x\|_2 = \left(\int_a^b [|x(t)|^2 + |x'(t)|^2 + |x''(t)|^2] dt \right)^{1/2}$

This norm is also referred to as the energy norm.

Convergence

In a normed linear vector space an infinite sequence of vectors $\{x_n\}$ is said to be convergent to a vector x if the sequence of real numbers $\|x - x_n\|$ converges to zero. In this case we write $x_n \to x$.

Transformation

Let X and Y be linear vector spaces, and let D be a subset of X. A rule, which associates with every element $x \in D$ a corresponding element $y \in Y$, is called a transformation from X to Y with domain D. This

transformation is illustrated in Figure 2. We write $y = T(x)$. The collection of all vectors $y \in Y$ for which there is an $x \in D$ with $y = T(x)$ is called the range R of T.

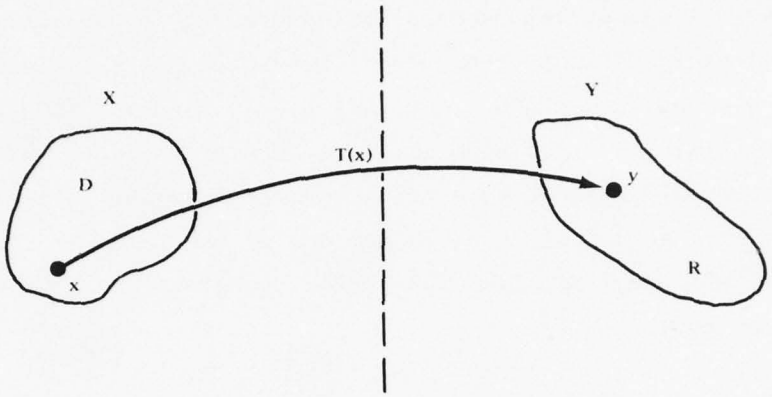

Figure 2. Transformation between two vector spaces.

Functional

A transformation from X into the space of real numbers (scalars) R^1 is called a functional of X, usually denoted f(x), g(x), etc. Examples of functionals include the following:

(a) $f(x) = \|x\|$

(b) $f(x) = \int_a^b x(t)dt$

The idea of a functional is central to the Rayleigh-Ritz technique and variational principles.

Cauchy Sequence

A sequence $\{x_n\}$ in a normed space is said to be a Cauchy sequence if $\|x_n - x_m\| \to 0$ as $m,n \to \infty$. In a normed space every convergent sequence is a Cauchy sequence since,

$$\|x_n - x_m\| = \|x_n - x + x - x_m\|$$

$$\leq \|x_n - x\| + \|x - x_m\| \to 0$$

However, a Cauchy sequence is not necessarily convergent. For example, $\{1/n\}$ is a Cauchy sequence in the space $X = R - 0$ which is the real line R minus the origin. But this Cauchy sequence is not convergent in X since $\{1/n\}$ converges to 0 which is outside the space X.

More generally, if we take any sequence of points in a vector space which converges to a limit that is not one of the terms in the sequence, and then delete the limit from the vector space, we get a Cauchy sequence which is not convergent.

Complete

A normed linear vector space is complete if every Cauchy sequence from X has a limit in X. That is, X must contain the limit as well as the members of the sequence. A complete normed linear vector space defines a Banach space.

Generally, it is advantageous to formulate problems in such a space because we can easily test sequences for convergence by asking whether they are Cauchy or not. If they are, we have assurance that the sequence is convergent since the space is a completed space. We do not have to know, a priori, what the limit is, and fortunately so, because its value is often the object of the problem.

Consider the space of rational numbers R'. Let $\{x_n\}$ be a sequence of rational numbers converging to $\sqrt{2}$. This sequence is Cauchy, but R' does not contain the limit element $\sqrt{2}$ ($\sqrt{2}$ is an irrational number). Therefore R' is not complete.

While the space of rational numbers is not complete, the space of real numbers R is complete. Note $\sqrt{2}$ is a real number, and thus R contains the sequence $\{x_n\}$ and the limit.

It is apparent by now that there may be more than one normed space of continuous functions. It depends only on the definition of the norm. To see this, consider the space of continuous functions on $[0,1]$. We can define two different normed spaces as follows:

(a) Let X be the space of continuous functions on $[0,1]$ with norm defined as

$$\| x \| = \int_0^1 |x(t)| \, dt$$

(b) Let X be as in (a) above except that the norm is defined as

$$\| x \| = \max_{0 \leq t \leq 1} |x(t)|$$

These are two different normed vector spaces, but the first is not complete while the second is.

The first is not complete because a sequence of functions in X is defined as follows

$$x_n(t) = \begin{cases} 0 & 0 \leq t \leq 1/2 - 1/n \\ nt - n/2 + 1 & 1/2 - 1/n \leq t \leq 1/2 \\ 1 & t \geq 1/2 \end{cases}$$

A graphical portrayal of this sequence and its convergence (lack of) is shown in Figure 3.

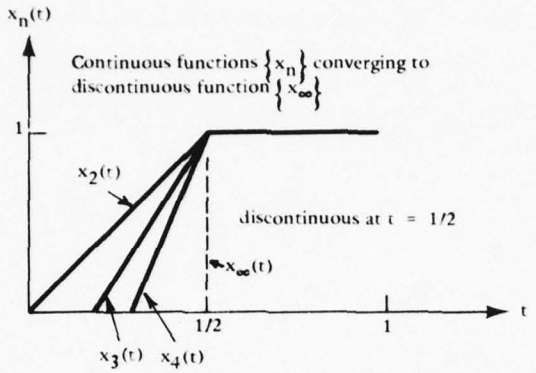

Figure 3. Convergence of a sequence of functions.

First it can be verified that $\{x_n\}$ is Cauchy since

$$\left\| x_n - x_m \right\| = 1/2 \left| 1/n - 1/m \right| \rightarrow 0$$

However, this sequence converges to a discontinuous function as shown, and therefore does not converge to a member of X. As a result, the normed vector space is not complete.

It can be shown that every Cauchy sequence in the second normed vector space does have a limit in the space. The main difference here is that the example sequence is not Cauchy with respect to the norm defining the space.

Inner Product

On the Cartesian product space X x X an inner product is defined as follows. Corresponding to each pair of vectors (x,y) in X the inner product $\langle x,y \rangle$ of x and y is a scalar. The inner product must satisfy the following four axioms in a real vector space X.

(1) $\langle x,y\rangle = \langle y,x\rangle$

(2) $\langle x + y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$

(3) $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$

(4) $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0$ if $x = \Theta$

The inner product is a generalization of the dot product in two- or three-dimensional Euclidean space.

Hilbert Space

As we have seen, a Banach space is a complete normed vector space. A Hilbert space is a Banach space plus an inner product which defines the norm; i.e., $\|x\| = \langle x,x\rangle^{1/2}$.

The key concept involved with a Hilbert space is orthogonality. This concept is not generally available in other spaces. Two vectors in a Hilbert space are defined as orthogonal if their inner product is zero. The methods of Ritz, Galerkin, and, of course, finite element have natural settings in the framework of Hilbert spaces, as do the concepts of optimization, Fourier series, least-squares minimization, and ortho-normal bases. Hilbert spaces then provide a unifying foundation for many areas of engineering.

Projection Theorem

The shortest distance from a point to a plane is a line given by the perpendicular from the point to the plane. This obvious and intuitive result can be generalized to the problem in any higher space (n-dimensional Euclidean space or a continuous function space) of finding the "shortest vector," as measured by a valid norm, from a point to a subspace. The approach seeks to find that vector which is orthogonal to the subspace. This is a brief introduction to the projection theorem.

Let H be a Hilbert space and M a closed subspace of H. Corresponding to any vector $x \in H$, there is a unique vector $m_o \in M$ such that $\|x - m_o\| \leq \|x - m\|$ for all $m \in M$. Furthermore, a necessary and sufficient condition that m_o be the unique minimizing vector is that $x - m_o$ be orthogonal to M. The three-dimensional version of the projection theorem is shown below in Figure 4.

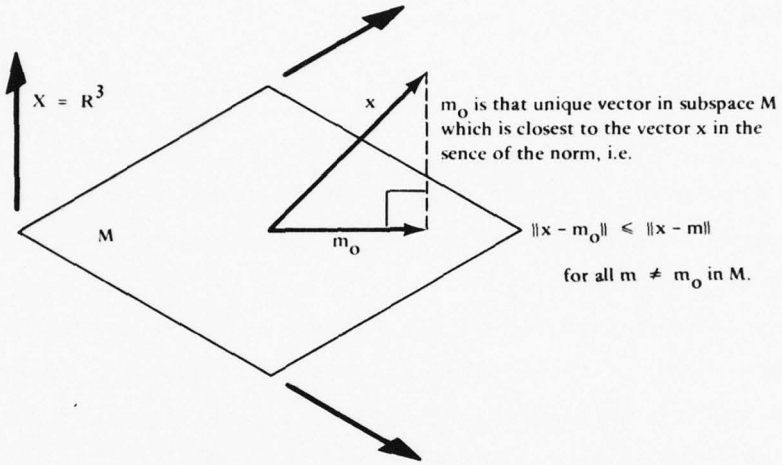

Figure 4. Projection theorem concept.

SOME FINITE ELEMENT CONCEPTS FROM MATHEMATICAL THEORY

If one is not interested in the mathematical questions of what constitutes valid shape functions, accuracy, convergence, and, in general, why the method is efficient in many problems, it would not be necessary to know little beyond the basic principles of virtual work and/or equilibrium to enable successful application of the method. Further, the basic reason that the finite element method is successful and efficient is that the polynomial shape functions which it employs are both computationally efficient and good mathematical approximations. The method does not owe its success to physical principles, though this sometimes

appears to be the case since these principles are the source of the all-important governing differential or integral equation. However, the finite element method allows an efficient discretization and numerical solution of the governing equations after they have been established. Before the ingenious application of the shape functions that are attributed to the finite element method, the discretization was often accomplished with classical Ritz functions. The discretization, however, remains fundamentally a Ritz process, and to understand the finite element method and the reason for its success, it is necessary to follow through the development of the Ritz method.

Approximate Method of Rayleigh-Ritz

There are a number of references on the Rayleigh-Ritz method, but a book by Mikhlin* is frequently noted in the literature of finite element. This reference was used in the preparation of the following material, and though it covers a good deal more than Rayleigh-Ritz it still is very complete on the subject.

Consider solving the time-independent differential equation

$$L\,u = f \tag{1}$$

in some domain Ω where L is some differential operator which acts upon an unknown function u to yield a known function f. On the boundary Γ of Ω, u must satisfy certain boundary conditions. The differential operator L is linear and defined for a subspace M of a real Hilbert space H. If L contains derivatives up to the mth order, then M is the space of all continuous functions having continuous mth order derivatives. The larger space H is taken as the space of all functions which are square summable and have the inner product

$$\langle u,v \rangle = \int_{\Omega} u\,v\,d\,\Omega \qquad u,v \in H$$

*S. G. Mikhlin. Variational methods in mathematical physics. Oxford, England, Pergamon Press, 1964.

This merely means that $\langle u,v \rangle < \infty$ and therefore the inner product will exist. (This is not a very restrictive definition at all, and therefore it is useful to think of H as being very large.) Let f also be a member of H so that Equation 1 defines a linear transformation or mapping as shown below in Figure 5.

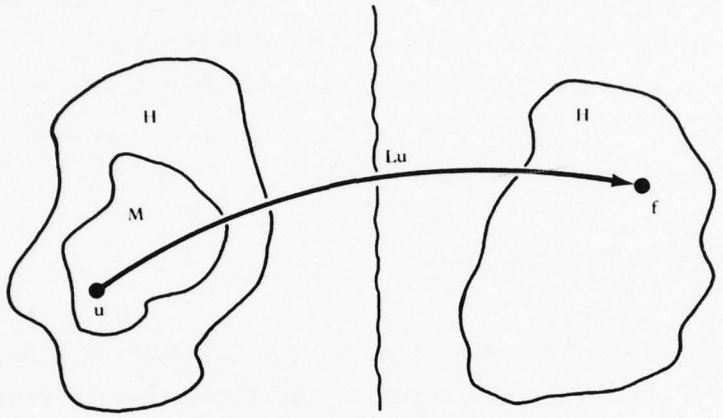

Figure 5. Linear transformation from subspace M to space H.

Let D_L be a subspace of M such that in addition to the requirements of being a member of M, the functions $u \in D_L$ must also satisfy the boundary conditions. The term D_L is called the field of definition of operator L. For the time being let it be stated that operator L must be positive definite; that is, $\langle Lu,u \rangle \geq 0$ for all $u \in D_L$ where the equality holds if, and only if, $u = \theta$ (θ being the null vector).

With the above definitions established we state, without proof, the following equivalent variational formulation of the boundary value problem. Finding the function u* which yields the minimum of the quadratic functional

$$F(u) = \langle Lu,u \rangle - 2\langle u,f \rangle \qquad (2)$$

is the same as solving the governing differential Equation 1 for u. A specific example of a variational formulation, common to structural

engineers, is the solution of the partial differential equation of static elasticity theory by, instead of integrating the differential equation, finding the minimum of the potential energy functional for the elastic body. A more concise statement of the equivalency is as follows. If, in addition to being positive definite, L is a symmetric ($\langle Lu,v\rangle = \langle Lv,u\rangle$) *operator, then if* $Lu = f$ is to have a solution in M, this solution is also the minimizing function for Equation 2. Conversely, if u^* minimizes $F(u)$ then $u^* = u$ and is the solution to $Lu = f$.

In solving the linear differential equation of Equation 1 it should be noted that while the given function f can be any function so long as it is a member of the Hilbert space H, the sought-after function u must be a member of the field of definition of the linear operator L as well as a member of H. Thus if L is a fourth order operator, u must be a continuous function having a continuous fourth order derivative and must satisfy certain boundary conditions to be a member of the field of definition D_L.

It can therefore occur that for some functions $f \in H$ there will not exist a function u in the field of definition that will satisfy the differential equation. As an example of this case, consider the problem of finding the deflection of a uniform cantilever beam under the action of a uniformly distributed load q(x). The appropriate equations are

$$E\,I\,y'''' = q(x) \quad x \in [0,L]$$

$$y(o) = y'(o) = y''(L) = y'''(L) = 0$$

The differential equation is derived by considering the equilibrium of an infinitesimal length of the beam under the assumption that the loading is continuous across the infinitesimal length.

In this case the operator is

$$L = E\,I\,\frac{d^4}{dx^4}$$

Its field of definition D_L is defined as the totality of those functions defined over $[0,L]$ which possess continuous fourth derivatives and which satisfy the boundary conditions of the problem. If $q(x)$ is continuous everywhere in $[0,L]$ there will exist a solution y in D_L. But if $q(x)$ is discontinuous, no solution can be found in D_L since y would have to possess a discontinuous fourth derivative to satisfy the governing equation. It should be noted that a discontinuous $q(x)$ is an entirely legitimate possibility because it would be square summable and therefore a member of H. In a practical sense a discontinuous $q(x)$ is very common.

This difficulty can be overcome by considering limits of functions that lie in D_L. Just as a discontinuous load $q(x)$ may be considered as the limit of a sequence of continuous loads, so functions with discontinuous fourth derivatives are introduced that are the limits of sequences of functions with continuous fourth derivatives. Extending the field of definition D_L in this manner can always be accomplished if L is a positive bounded below operator in addition to being symmetric and positive definite. The criterion for being positive bounded below is $\langle Lu,u\rangle \geq \gamma^2\langle u,u\rangle$ where $u \in D_L$ and γ^2 is any positive constant. It can then be asserted that among this new set of functions lies the solution of the differential equation for any $f \in H$.

This spatial extension creates a new Hilbert space H_L with an inner product $[u,v]$ where $u,v \in H_L$, and this inner product is sometimes called the energy product. It is defined as follows,

$$[u,v] = \lim_{n\to\infty} \langle Lu_n, v_n\rangle = \langle Lu, v\rangle$$

where u_n and v_n are sequences and $u_n, v_n \in D_L$. Note that through an integration by parts we get a new operator R as follows,

$$\langle Lu_n, v_n\rangle = \int_\Omega \left(Lu_n \cdot v_n\right) d\Omega = \int_\Omega \left(Ru_n \cdot Rv_n\right) d\Omega.$$

Thus,

$$[u,v] = \lim_{n\to\infty} \int_{\Omega} (Ru_n \cdot Rv_n)\, d\Omega = \lim_{n\to\infty} \int_{\Omega} (Rv_n \cdot Ru_n)\, d\Omega$$

$$= \lim_{n\to\infty} \langle Rv_n, Ru_n \rangle = [v,u],$$

and the energy product is shown to be symmetrical.

The Hilbert space H_L is important in the theory of finite element. The norm in H_L is the energy norm and is defined as

$$\| u \| = [u,u]^{1/2} = \left(\int_{\Omega} (Ru \cdot Ru)\, d\Omega \right)^{1/2}.$$

The set of admissible functions in H_L is given by considering sequences of functions, say u_n, that satisfy

$$\lim_{n,m\to\infty} \| u_n - u_m \| = 0.$$

This implies the existence of an element u with finite norm to which the sequence u_n converges in the mean. This set of functions constitutes H_L.

Those functions constituting D_L must satisfy all the boundary conditions. But some functions in H_L which contains D_L often do not satisfy all the boundary conditions. Those boundary conditions which are satisfied by functions in D_L, but which are not satisfied by functions in H_L are called <u>natural</u> boundary conditions. Those boundary conditions which remain, or those boundary conditions which are satisfied by functions only in H_L are the <u>essential</u> (forced, principal) boundary conditions. The latter are required to assure that the energy norm is positive definite; i.e., $\| u \| = \sqrt{[u,u]} \geq 0$.

Now we return to the problem of minimizing F(u) where

$$F(u) = [u,u] - 2\langle u,f\rangle \tag{2a}$$

and where the candidate functions u are now members of H_L. The central result in going from Equation 2 to 2a is that now the trial functions u need satisfy a fewer number of boundary conditions i.e., the essential boundary conditions.

Following the classical Ritz method to approximate the minimizing function of F(u) we select a solution of the form

$$u^M = \sum_{k=1}^{M} a_k \phi_k^M, \tag{3}$$

where ϕ_k^M are the Ritz functions, and a_k are the unknown Ritz coefficients which are to be determined. If the functions ϕ_k^M, k=1,2,...,M, are complete in H_L then convergence to the correct answer u is assured in the sense of convergence of the energy norm. The ϕ_k^M are complete in H_L if for any $v \in H_L$ it is possible to choose $\varepsilon > 0$ and find an integer N and constants $a_1, a_2, \ldots, a_M$ such that

$$\left\| v - u^M \right\| < \varepsilon \text{ for } M > N$$

Substituting Equation 3 into Equation 2a, the quadratic functional becomes

$$F(u^M) = \left[\sum_{k=1}^{M} a_k \phi_k^M, \sum_{k=1}^{M} a_k \phi_k^M \right] - 2\left\langle \sum_{k=1}^{M} a_k \phi_k^M, f \right\rangle \tag{4}$$

or,

$$F(u^M) = \int_\Omega L \left(\sum_{k=1}^{M} a_k \phi_k^M \right) \sum_{k=1}^{M} a_k \phi_k^M \, d\Omega \qquad (4a)$$

$$- 2 \int_\Omega \left(\sum_{k=1}^{M} a_k \phi_k^M \right) f \, d\Omega$$

To find the a_k, $k=1,2,\ldots,M$, corresponding to the minimum of $F(u^M)$, we differentiate Equation 4a with respect to a_j and set the result to zero.

$$\frac{\partial F}{\partial a_j} = \int_\Omega \left[L \left(\phi_j^M \right) \sum_{k=1}^{M} a_k \phi_k^M + L \left(\sum_{k=1}^{M} a_k \phi_k^M \right) \phi_j^M \right] d\Omega \qquad (5)$$

$$- 2 \int_\Omega \phi_j^M f \, d\Omega = 0$$

or

$$\left[\phi_j^M, \sum_{k=1}^{M} a_k \phi_k^M \right] + \left[\sum_{k=1}^{M} a_k \phi_k^M, \phi_j^M \right] - 2 \langle \phi_j^M, f \rangle = 0 \qquad (5a)$$

Because the energy product is symmetrical, this equation reduces to

$$\left[\phi_j^M, \sum_{k=1}^{M} a_k \phi_k^M \right] - \langle \phi_j^M, f \rangle = 0 \qquad (6)$$

Finally, Equation 6 is rearranged as follows,

$$\int_\Omega L \left(\phi_j^M \right) \sum_{k=1}^{M} a_k \phi_k^M \, d\Omega - \langle \phi_j^M, f \rangle = 0$$

or,

$$\sum_{k=1}^{M} a_k \left[\phi_j^M, \phi_k^M\right] = \langle \phi_j^M, f \rangle \quad j = 1,2,\ldots,M \tag{7}$$

Thus, the end result of the Ritz process is Equation 7 which is a linear system of M algebraic equations from which we can solve for the a_k of Equation 3.

Advantage of Finite Element Shape Functions

Strang and Fix provides an excellent discussion on the subject of the mathematical theory of finite element.* The authors, who are mathematicians, have learned the "language of finite element," and it is evident in their narrative. However, their treatment does assume that readers are comfortable with many of the functional analysis concepts presented earlier in this report. Most engineers who practice finite element may not, however, be knowledgeable in this area of mathematics. Nevertheless, the purpose of this section and a main point of this report can be satisfactorily understood without thoroughly understanding functional analysis. The authors contend that shape functions are a root-cause of the success of the finite element method. This contention is developed here in the context of the earlier introductions to functional analysis and the Rayleigh-Ritz method.

It was stated earlier that the main reason for the success of the finite element method was its computational ease and accuracy and that this was a result of using certain kinds of functions called shape functions for the ϕ_k^M in the Ritz process. This will be shown in the following examples by using three different kinds of functions for the solution of a boundary value problem, and, in each case, taking the computation as far as Equation 7.

*G. Strang and G. J. Fix. An analysis of the finite element method. Englewood Cliffs, N.J., Prentice-Hall, Inc., 1973.

The statement of the illustrative problem is as follows:

$$L\,u = f,$$

where

$$L = \frac{d^2}{dx^2} + 1$$

and

$$f = x.$$

The boundary conditions are $u(0) = u(1) = 0$, and the domain is $\Omega = [0,1]$.

We assume a solution of the form

$$u^M = \sum_{k=1}^{M} a_k \phi_k^M$$

Case 1. Let the Ritz functions be the eigenfunctions for the given differential equation; i.e.,

$$\phi_k^M = \sin k\pi x.$$

These functions are shown in Figure 6 and are seen to satisfy the given boundary conditions.

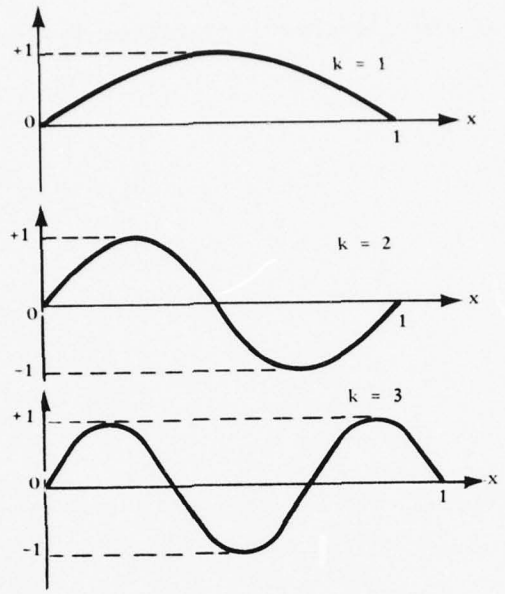

Figure 6. Eigenfunctions used as Ritz functions.

Substituting L, f, and ϕ_k^M into Equation 7, we get

$$\sum_{k=1}^{M} a_k \int_0^1 \left[\frac{d^2}{dx^2} (\sin j\pi x) + \sin j\pi x\right] \sin k\pi x \, dx = \langle \phi_j, x \rangle$$

$$j = 1,2,\ldots,M.$$

This equation reduces successively as follows,

$$\sum_{k=1}^{M} a_k \int_0^1 \left[1 - (j\pi)^2\right] \sin j\pi x \sin k\pi x \, dx = \langle \phi_j, x \rangle$$

$$j = 1,2,\ldots,M,$$

or

$$\left[1 - (j\pi)^2\right] \sum_{k=1}^{M} a_k \int_0^1 \sin j\pi x \sin k\pi x \, dx = \langle \phi_j, x \rangle$$

$$j = 1,2,\ldots,M.$$

Invoking orthogonality of the eigenfunctions, $\langle \phi_j, \phi_k \rangle = \delta_{jk}$, the above equation reduces to

$$[1 - (j\pi)^2] \sum_{k=1}^{M} a_k \delta_{jk} = \langle \phi_j, x \rangle \qquad j = 1,2,\ldots,M$$

or

$$[1 - (j\pi)^2] a_j = \langle \phi_j, x \rangle \qquad j = 1,2,\ldots,M.$$

That is, Equation 7 reduces to the following system of equations in the unknown a_j

$$\begin{bmatrix} 1 - \pi^2 & & & \\ & 1 - 4\pi^2 & & \\ & & 1 - 9\pi^2 & \\ & & & \ddots \\ & & & 1 - M^2\pi^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_M \end{Bmatrix} = \begin{Bmatrix} \langle \phi_1, x \rangle \\ \langle \phi_2, x \rangle \\ \langle \phi_3, x \rangle \\ \vdots \\ \langle \phi_M, x \rangle \end{Bmatrix}$$

Thus, the use of eigenfunctions for the shape functions results in a <u>diagonal</u> coefficient matrix. The computational ease of solving for a_j is optimal in this case. But the eigenfunctions are not, of course, generally available a priori.

Note also that the approximate solution

$$u^M = \sum_{k=1}^{M} \frac{\langle \phi_k, x \rangle}{1 - k^2\pi^2} \sin k\pi x$$

can be thought of as the projection of the true solution,

$$u = \sum_{k=1}^{\infty} \frac{\langle \phi_k, x \rangle}{1 - k^2\pi^2} \sin k\pi x,$$

onto the subspace M spanned by the first M eigenfunctions.

Case 2. Let the Ritz functions be the polynomials,

$$\phi_k^M = x^k (1 - x)$$

and note that they, too, satisfy the boundary conditions.

The first three of these polynomial functions are shown in Figure 7.

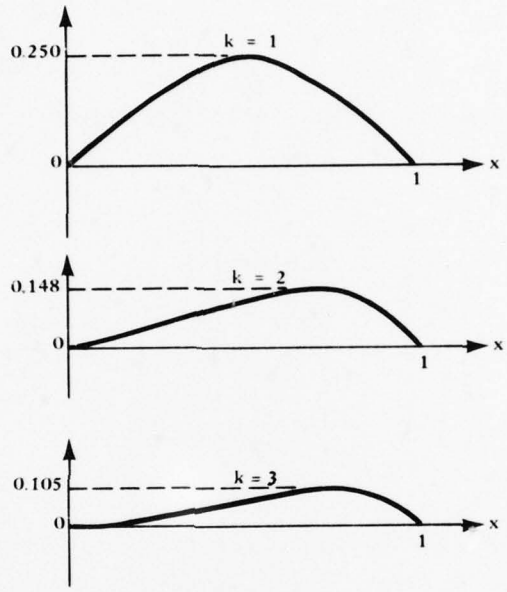

Figure 7. Polynomials used as Ritz functions.

Substituting these polynomials into Equation 7 we get

$$\sum_{k=1}^{M} a_k \int_0^1 \left[\frac{d^2}{dx^2} x^j(1-x) + x^j(1-x) \right] x^k(1-x)dx$$

$$= \langle x^j(1-x), x \rangle \qquad j = 1,2,\ldots,M$$

or

$$\sum_{k=1}^{M} a_k \int_0^1 \left[j(j-1)x^{j-2}(1-x) - 2jx^{j-1} + x^j(1-x) \right] x^k(1-x)dx$$

$$= \langle x^j(1-x), x \rangle \quad j = 1,2,\ldots,M$$

Entries in the first row of the coefficient matrix, obtained by setting $j = 1$, are

$$\int_0^1 \left[-2 + x(1-x) \right] x(1-x)\, dx$$

$$\int_0^1 \left[-2 + x(1-x) \right] x^2 (1-x)dx, \ldots$$

$$\int_0^1 \left[-2 + x(1-x) \right] x^M (1-x)dx$$

Elements of the second row, obtained by setting $j = 2$, are;

$$\int_0^1 \left[2(1-x) - 4x + x^2(1-x) \right] x\,(1-x)dx$$

$$\int_0^1 \left[2(1-x) - 4x + x^2(1-x) \right] x^2\,(1-x)dx, \ldots$$

$$\int_0^1 \left[2(1-x) - 4x + x^2(1-x) \right] x^M\,(1-x)dx$$

Thus, the coefficient matrix is not diagonal, and with a little more study it becomes apparent that it is full. For example, if we take $M = 2$ and write out the linear system we get

$$\begin{bmatrix} 3/10 & 3/20 \\ 3/20 & 13/105 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 1/12 \\ 1/20 \end{Bmatrix}$$

Solving this system yields, $a_1 = 71/369$, and $a_2 = 7/41$. Therefore, the approximate solution for M = 2 is,

$$u^M = 71/369\ x(1-x) + 7/41\ x^2(1-x).$$

Case 3. Let ϕ_k^M be the finite element hat-shaped functions defined as follows,

$$\phi_k^M = \begin{cases} 0 & 0 \leqq x \leqq (k-1)\,h \\ 1/h\ x - (k-1) & (k-1)\,h \leqq x \leqq kh \\ -1/h\ x + (k+1) & kh \leqq x \leqq (k+1)\,h \\ 0 & (k+1)\,h \leqq x \leqq 1 \end{cases}$$

Three consecutive shape functions are graphed in Figure 8, where it can be seen that they satisfy the boundary conditions. Further, they are continuous functions over the entire region [0,1]. They do not differ in this respect from the eigenfunctions and polynomial functions which were employed in the previous two cases.

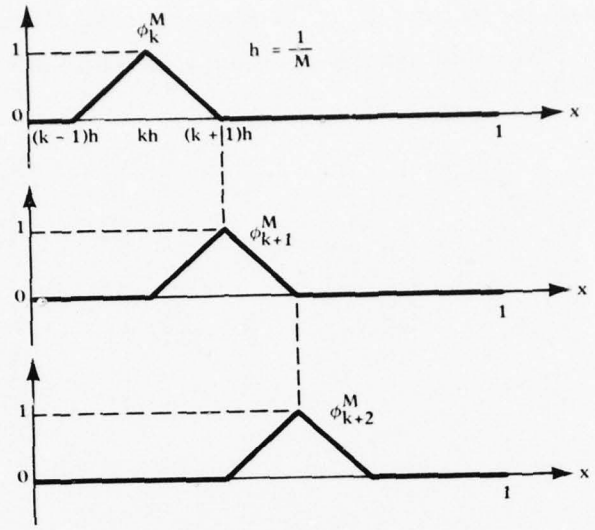

Figure 8. Finite element shape functions used as Ritz functions.

They do differ in that at x = kh, for example, one function equals unity, while all others equal zero. Such points are termed node points, and there the undetermined coefficients become the value of the dependent variable u(kh). Substituting the shape functions into Equation 7 we get,

$$\sum_{k=1}^{M} a_k \left[\int_0^1 (\phi_j^{M''} + \phi_j^M)\, \phi_k^M\, dx\right] = \langle \phi_j^M, x \rangle \qquad j = 1,2,\ldots,M.$$

In this case, however, $\phi_j^{M''} = 0$, and therefore this equation reduces to,

$$\sum_{k=1}^{M} a_k \left[\int_0^1 \phi_j^M\, \phi_k^M\, dx\right] = \langle \phi_j^M, x \rangle \qquad j = 1,2,\ldots,M.$$

We note that the product $\phi_j^M\, \phi_k^M$ will be zero unless $|j - k| \leq 1$. This can best be seen from the above sketches of the shape functions. For

example, the product of ϕ_k^M and ϕ_{k+1}^M is nonzero, but the product of ϕ_k^M and ϕ_{k+2}^M is zero. Thus the system of equations for Case 3 is <u>tridiagonal</u> as follows,

$$\begin{bmatrix} \left[\phi_1^M,\phi_1^M\right] & \left[\phi_1^M,\phi_2^M\right] & & \\ \left[\phi_2^M,\phi_1^M\right] & \left[\phi_2^M,\phi_2^M\right] & \left[\phi_2^M,\phi_3^M\right] & \\ & \left[\phi_3^M,\phi_2^M\right] & \left[\phi_3^M,\phi_3^M\right] & \left[\phi_3^M,\phi_4^M\right] \\ & & & \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} \langle \phi_1^M, x \rangle \\ \langle \phi_2^M, x \rangle \\ \langle \phi_3^M, x \rangle \\ \vdots \end{Bmatrix}$$

The primary observation is that the system is almost diagonal; or, alternatively, orthogonality almost exists. Computationally, the system of equations resulting from the use of finite element shape functions is preferable to the system resulting from the use of Ritz functions (smooth polynomials). Surprisingly, the only difference between the Ritz and the finite element method is in the choice of the continuous functions ϕ_k^M for use in the approximate solution

$$u^M = \sum_{k=1}^{M} a_k \phi_k^M.$$

In each of the three cases the ϕ_k^M are defined continuously over the entire region $\Omega = [0,1]$ and $\phi_k^M \in H_L$. However, the differences are important and are summarized as follows:

(a) The finite element functions ϕ_k^M (Case 3) are zero everywhere except over a subdomain or element. This reduces the coupling in the linear system given by Equation 7.

(b) The finite element functions ϕ_k^M are defined such that they are unity at $x = jh$, $j = 1,2,\ldots,M$, that is at the node points. This assigns physical significance to the a_k. That is, $a_k = u^M(kh)$, and these are the values of the dependent variable at the kth node points.

SUMMARY

The problem at hand is generally one of finding an approximate solution to a given boundary value problem; the solution must satisfy a governing differential equation and its associated boundary conditions. The process of finding an approximate solution can be discussed in a heuristic way, by relying on geometrical notions. Though we actually search for solutions within mathematically complex subspaces of continuous functions, often from a practitioner's point of view, significant insight is gained by limiting thought to Euclidian space. It is helpful to imagine that the "true" solution is a vector imbedded in three-dimensional space, but that the search for an approximate solution vector is unfortunately limited to some lesser subspace (say, a plane). The projection theorem provides the interpretation that the best approximate solution within that restriction is an orthogonal projection of the true solution vector onto the lesser subspace. The Ritz (or, more generally, the Galerkin) method is the mechanism for finding that projection.

The error vector is the difference between the true and approximate solution vectors and is normal to the subspace when it is minimized. It is measured by the energy norm. It is the business of theoreticians to obtain estimates of the energy norm of the error vector for formulations so that the relative accuracy and convergence of alternatives can be assessed. Accuracy criterion is based on the energy norm, and convergence occurs, as is often written, "in the sense of the norm." The energy norm is a functional; as such, it is a particular type of transformation. Its evaluation yields a real number when it is provided with an input,

continuous function. Thus, it transforms an element of continuous function space into an element of real number space. In this context, the continuous functions are finite element shape functions, and the energy norm can be regarded as a mathematical "measuring stick" for such functions.

The subspace of admissible functions in which an approximate solution is sought is dictated by the differential operator's order and the boundary conditions. It is constituted of functions having continuous derivatives up to the order of the operator and that also satisfy all the boundary conditions, plus some additional functions. These additional, admissible functions are defined by an extension of the field of definition for the operator. They are all functions that are obtainable as limits of sequences of admissible functions from the first group, the field of definition. A significant practical benefit requires mentioning here. Often, these limit functions possess lower order differentiation than the functions constituting the field of definition. Because of this, they are easier to construct and simpler to use in subsequent computations. Further, due to their reduced order of differentiation, they do not, and need not, satisfy boundary conditions which generally involve higher order derivatives on the dependent variable; boundary conditions that they do not satisfy are the natural boundary conditions. Those boundary conditions that they do satisfy are the essential boundary conditions. Construction of functions from the field of definition is most often prohibitively difficult and/or they may be too costly to compute with. However, if they are available and economical, their use is preferable and recommended on grounds of increased accuracy.

Finite element shape functions are continuous over the entire region and are no different than classical Ritz functions in this regard. They do differ in that they are piecewise continuous, each being defined as zero everywhere except over a subdomain (element) of the region. On the other hand, classical Ritz functions are generally more smooth and nonzero everywhere over the region. A system of linear, algebraic equations that results from using piecewise continuous functions is

solved more easily than a system that results from Ritz functions. This is because they are characteristically banded. The bandedness reflects local coupling among the equations written for individual subdomains in proximity, whereas global coupling and full matrices are inherent with the Ritz functions. In either case, the undetermined coefficients in the Ritz process are found from solution of the linear system of equations. They have a more useful interpretation, though, when finite element shape functions are used. In this case, the coefficients are the values of the dependent variable at node points, and are therefore directly useable data. That is, they are often discrete values of the solution sought. In contrast, the coefficients of classical Ritz functions cannot be interpreted meaningfully. Useable data are obtainable only with further manipulation of the coefficients after their numerical values are found.

The finite element method, in large measure, owes its success to piecewise continuous shape functions. It remains fundamentally a Ritz (more generally, Galerkin) approximation process, but because shape functions are simple and amenable with the digital computer, the finite element method has transcended the classical method of Ritz in application and popularity. Nevertheless, it is instructive to understand the Ritz process as a method for minimizing a functional. The function, within a prescribed space of admissible functions, that minimizes the functional is found by the Ritz process and is also an approximate solution to the boundary value problem. A certain amount of mathematics from functional analysis is required to appreciate the Ritz process, and the finite element method as well. It begins with the definition of linear vector spaces and proceeds through to the definition of Hilbert spaces. The mathematics can be presented for practitioners by relying, where possible, on geometric interpretation in more familiar Euclidean space.

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